New Advanced Mathematics

A Complete HSC Mathematics Extension 2 Course

Terry Lee
B.Sc., M.Ed.

New Advanced Mathematics

A COMPLETE HSC MATHEMATICS EXTENSION 2 COURSE

Terry Lee

B.Sc., M.Ed.

NEW ADVANCED MATHEMATICS

A Complete HSC Mathematics Extension 2 Course

First edition.
ISBN: 978-09577428-8-8
Published: March 2019

Visit our website to buy this book online or to download free of charge Terry's solutions to past HSC papers.
advancedmathematics.com.au

Printed in Australia

Table of Contents

Preface

This book is written for the new Mathematics Extension 2 course, which is being introduced into the NSW syllabus in 2020.

This book has been written with two main objectives: it can be used as a textbook for classroom use, as well as a step-by-step resource to be used independently by students for their own self-study purposes. This book provides sufficiently clear explanations about each topic in the syllabus, with worked out examples and alternative methods, where applicable. Questions are categorised by topic and graded from easy to hard, to help guide students in their learning. Each chapter also contains a set of review exercises and challenge problems, as well as fully worked solutions for each question. The review exercises will help consolidate students' skills and knowledge, while improving their competence and confidence. The book also features challenge problems. While they may extend beyond the syllabus, they are designed to provide extra stimulus for highly motivated students and increase confidence for the harder questions in the Higher School Certificate examination.

While this book is written specifically for the new Mathematics Extension 2 course, it also covers various topics in the Advanced Mathematics and Mathematics Extension 1 courses, including Mathematical Induction, Vectors, Integration and Elementary Dynamics. This book provides clear explanations of the basic concepts underpinning these topics to help provide students with a strong foundation upon which they can develop a more thorough understanding of the complex and challenging concepts in Mathematics Extension 2.

This book also features colour-coding throughout to highlight various theorems and study tips – this makes the book a study reference and more enjoyable to read. Students are advised to complete as many questions in this book as possible to master the course.

This book builds upon what the Terry Lee series has been famous for: it includes many fully explained tips and tricks to help students understand and solve problems efficiently, while ultimately developing a greater enjoyment of the course. The joy in being able to complete Mathematics Extension 2 questions, including harder questions, in half the time is immeasurably worthwhile.

Terry Lee

1 The nature of proof

HSC Outcomes
A student
understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts
chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings
applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

In this chapter,

1.1 The nature of proof

Mathematicians accept the truth of nothing without proof. A mathematical statement is either true or false, but not both. To prove the truth of a statement is to provide a logical reason or a finite sequence of logical reasons to support its truth. There are several ways to do so. But before we look at methods of proof, let's learn about the language of proof.

1.1.1 The language of proof

A good proof must not only use logical reasons correctly, but also the language correctly.

(a) It uses conventions, e.g. given two points *A* and *B*, $\overrightarrow{AB}$ is the vector from *A* to *B* (not the other way around), m_{AB} is the gradient of the line *AB*[1], but *AB* could be the line *AB* or the interval *AB* or the distance *AB*. To tell the difference of these usages, one must consider the context of the sentence being used. For example, if you are asked to (1) prove that $AB \perp CD$, you must understand that *AB* and *CD* are two lines, and the intervals *AB* and *CD* may not intersect and (2) find the ratio of *AC*:*AB* if *C* is a point that divides *AB* in the ratio 1:2 internally then both *AC* and *AB* are distances.

(b) It uses symbols. The following symbols are commonly used.
(i) In set theory, $\in$ (belong to), $\cup$ (union), $\cap$ (intersection), $\subset$ (subset). Also, *N* = the set of natural numbers[2], *J* or *Z* = the set of integers[3], *Q* = the set of rational numbers, *R* = the set of real numbers and *C* = the set of complex numbers.
(ii) In writing, $\Rightarrow$ (implies), $\Leftrightarrow$ (equivalent to), $\equiv$ (congruent), $|||$ (similar), $\therefore$ (therefore), $:$ (so that), $\because$ (since), iff (if and only if), $\forall$ (for all) and $\exists$ (there exists).
For example, "$2^n > n^2,\ \forall n \ge 5, n \in Z$" means $2^n > n^2$ for all values of $n \ge 5$, where n is an integer; "$\exists n \in N : 2^n = n^2$" means there exists a positive integer *n* such that $2^n = n^2$.

(c) It uses words that may not have the same meaning as in our commonly used language. For example, you may have asked 'what is the combination of this safe?', when you wanted to refer to the permutation of the safe. Can you answer this question? When a die is tossed, what is the probability that the upward face shows numbers '1' and '2'? The answer is 0, as it cannot show both numbers. If the above question changes 'and' to 'or', then the answer would be one third.

(d) The converse of a statement is not always true. The converse of 'if *A* then *B*' is 'if *B* then *A*', where *A* and *B* are statements. For example, 'if it rained last night then the grass is wet today' does not imply that 'if the grass is wet today then it rained last night'.

---

[1] Why is *m* the symbol for gradient? No one knows, although in CRC Concise Encyclopedia of Mathematics, Eric Weisstein claimed that it was first used in 1844 by British mathematician Matthew O'Brien. Some of you may like this explanation of the historian Howard Eves: 'because the word *slope* starts with the letter *m*' (Mathematical circles revisited, 1971, page 142).

[2] Is 0 a natural number? Natural numbers are counting numbers, they are 1, 2, 3, …, and whole numbers are 0, 1, 2, 3, …. Thus, 0 is a whole number, not a natural number.

Is 0 odd or even? Encyclopedia Britannica argues that as 0 is a number without remainder, it classifies as an integer. Therefore, as 2*k* is even and 2*k* + 1 is odd, where $k \in Z$, 0 is even.

Can odd or even numbers be negative? Yes, the definition that '2*k* is even and 2*k* + 1 is odd' applies for $k \in Z$.

[3] Why is *Z* used for integers? *Z* stands for the German word *zahlen*, which means integers. Apparently, as *I* is not a good choice for integers (in English, I is a pronoun), *Z* and the next letter *J* are chosen.
Z was first used in Grundlagen der Analysis, 1930, by Edmund Landau. We did not know when *J* was first used, but *J* was mentioned in Survey of Modern Algebra, 1953, by Garrett Birkhoff and Saunders MacLane.

Correctly, these set symbols should be written in double-struck style as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, but they are kept in simple italic style in this book.

There are different ways to prove a statement.

1.1.2 Direct proof

Example 1.1

If n is an odd integer, prove that n^2 is also an odd integer.

Let $n = 2k+1$, where $k \in Z$.

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$.

As $4k^2$ and $4k$ are both even, $4k^2 + 4k + 1$ is an odd integer, $\therefore n^2$ is odd.[4]

1.1.3 Proof by contradiction.

Example 1.2

Prove that $\sqrt{2}$ is an irrational number.

Assume that $\sqrt{2}$ is rational. Let $\sqrt{2} = \dfrac{p}{q}$, where p and q are positive integers and have no common factor.

$2 = \dfrac{p^2}{q^2}$, on squaring both sides.

$p^2 = 2q^2$. (1)

$\therefore p^2$ is even, $\therefore p$ is even.

Let $p = 2k$, where k is an integer.

Substituting to (1), $4k^2 = 2q^2$.

$\therefore q^2 = 2k^2$.

$\therefore q^2$ is even, $\therefore q$ is even.

But p and q assumingly do not have any common factors, they cannot both be even. $\therefore$ The assumption that $\sqrt{2}$ is rational is incorrect. $\therefore \sqrt{2}$ is irrational.

1.1.4 Proof by contrapositive

A statement and its contrapositive are equivalent.

The contrapositive of 'if A then B' is 'if not B then not A', where A and B are statements.
For example, 'if there is a blackout then the light must be off' is the same as 'if the light is on then there is no blackout'. Note: It does not necessarily mean that if there is no blackout then the light is on.

Use this method if a direct proof is not easy.

Example 1.3

If n^2 is an even integer prove that n is even.

Assume n is not an even integer, i.e. an odd integer, then since the product of 2 odd numbers is an odd number, n^2 is odd, i.e. n^2 is not even.

Therefore, by contrapositive, if n^2 is even then n is even.

---

[4] Some authors prefer to write QED to indicate that the proof is complete. QED is short for Latin 'Quod erat demonstrandum', which is a translation of 'that which was to be demonstrated'. This book chooses not to use it.

1.1.5 Proof by counter-examples

Example 1.4

Prove that the statement "$2^n - 1$, for $\forall n \geq 2, n \in Z$, is a prime number" is not true.

Since the statement refers to all values of n such that $n \geq 2, n \in Z$, we only need to find just one counter-example to declare that the statement is not true.

By trial and error, let $n = 11, 2^{11} - 1 = 2047 = 23 \times 89$.

Therefore, $2^n - 1$, for $\forall n \geq 2, n \in Z$, is not a prime number.

Induction proofs and inequality proofs will be considered in the next two sections.

Exercise 1.1

1 Fill in the blanks with the most correct symbols, using these symbols $\forall, \exists, \Rightarrow$ and $\Leftrightarrow$. Briefly explain your choice.

(a) He scored 3 out of 10 __ he failed the test. (b) ___ even numbers are divisible by 4.

(c) $x^2 - 2x + 3 \geq 0$ ___ $x \in R$. (d) $y = e^x$ ___ $x = \log_e y$.

2 True or false? Justify your answers.

(a) $y = \sin x \Leftrightarrow x = \sin^{-1} y$. (b) $\sin x \cos x \leq 1, \forall x \in R$. (c) $A^2 = B^2 \Rightarrow A = B$.

(d) $AB // CD, AB = 3 \Rightarrow CD = 3$. (e) $\exists x \in R : \sqrt{x^2 + y^2} = x + y$. (f) $(A \cap B) \cup C = A \cap (B \cup C)$.

(g) Lines $Ax + By = C$ and $Dx + Ey = F, A, B, D, E \neq 0$, are $\perp$ iff $AD + BE = 0$ and // iff $AE = BD$.

3 Directly prove the following.

(a) Prove that the angle bisectors of the base angles in an isosceles triangle are equal.

(b) Prove that the square of any integer either is divisible by 3 or gives a remainder of 1 when it is divided by 3. Hint: Any integer is either divisible by 3 or not.

(c) If $x, y \in R$, prove that $|x + y| \leq |x| + |y|$ algebraically and interpret it geometrically. Consider 4 cases (i) $x \geq 0, y \geq 0$, (ii) $x < 0, y < 0$, (iii) $x > 0, y < 0$ and $x + y \geq 0$ (iv) $x > 0, y < 0$ and $x + y < 0$

(d) $\forall \theta \in R$, prove that $\sec^2 \theta + \operatorname{cosec}^2 \theta \geq 4$. (e) Prove that $2 < \left(1 + \frac{1}{n}\right)^n < 3, \forall n \geq 2, n \in N$.

Hint for part (e): Use $(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots + x^n$.

4 Prove by contradiction.

(a) Prove that $\sqrt{3}$ is irrational. (b) Prove that $\sqrt{6}$ is irrational.

(c) Prove that $\log_2 5$ is irrational. (d) Prove that $\log_3 5$ is irrational.

(e) If $A \cup B = A \cap B$, where A and B are two non-empty sets, prove that $A = B$.

(f) In question 3, part (a), we have asked you to directly prove that the angle bisectors of the base angles in an isosceles triangle are equal. But it is not so easy to prove the converse. Prove by contradiction that if two angle bisectors of a triangle are equal, prove that the triangle is isosceles.

5 Prove by contrapositive.

(a) Suppose $x \in Z$, if x^3 is even, prove that x is even.

(b) Suppose $x, y \in Z$, if xy and $x - y$ are even, prove that x and y are both even.

(c) Suppose $x, y \in R$, if $3(x - y)$ is even, prove that x and y have the same parity.

(d) Suppose $x \in Z$, if x^2+3x+1 is even, prove that x is odd.

(e) Suppose $n \in N$, if 2^n-1 is a prime number then n is a prime number.

6 Prove that the following statements are not true by counter-examples.

(a) $\forall n \in N, 3^n > n^3$.

(b) $\forall a,b>0, \sqrt{a^2+b^2}=a+b$.

(c) A quadrilateral is formed by joining any 4 points in a plane.

(d) The point of inflexion of a curve $y=f(x)$ corresponds to the point where $f''(x)=0$.

7 Find the mistakes in the following 'proofs'.

(a) Question: Prove that 1 = 2.

Let $x=y$,

$x+y=2y$, on adding y to each side

$x+y-2x=2y-2x$, on subtracting $2x$ to each side

$y-x=2(y-x)$

$1=2$, on dividing by $y-x$

(b) Question: Prove that 2 = 3.

$4-10=9-15$

$4-10+\frac{25}{4}=9-15+\frac{25}{4}$, on adding $\frac{25}{4}$ to each side

$\left(2-\frac{5}{2}\right)^2=\left(3-\frac{5}{2}\right)^2$, using $a^2-2ab+b^2=(a-b)^2$

$2-\frac{5}{2}=3-\frac{5}{2}$, on taking square roots

$\therefore 2=3$

(c) Question: Prove that 1 = 2 = 3 = 4 = …

$x^2+2x+1=(x+1)^2$.

$$x^2+2x+1-2(x+1)\left(x+\frac{1}{2}\right)+\left(x+\frac{1}{2}\right)^2=(x+1)^2-2(x+1)\left(x+\frac{1}{2}\right)+\left(x+\frac{1}{2}\right)^2.$$

$$\text{LHS}=x^2+2x+1-2x^2-3x-1+\left(x+\frac{1}{2}\right)^2$$

$$=x^2-2x^2-x+\left(x+\frac{1}{2}\right)^2$$

$$=x^2-2x\left(x+\frac{1}{2}\right)+\left(x+\frac{1}{2}\right)^2$$

$$=\left(x-\left(x+\frac{1}{2}\right)\right)^2.$$

$$\text{RHS}=\left((x+1)-\left(x+\frac{1}{2}\right)\right)^2.$$

$$\text{LHS}=\text{RHS}, \therefore x-\left(x+\frac{1}{2}\right)=(x+1)-\left(x+\frac{1}{2}\right).$$

$\therefore x=x+1$.

Thus, when $x=1$ we have $1=2$, when $x=2$ we have $2=3$, and so on, $\therefore 1=2=3=4=...$

1.2 Induction proofs

The term "Induction" simply means the formation of a suitable formula suggested from a number of observations. For example, after many unfortunate incidents had occurred on Friday 13th, people came to the conclusion that Friday 13th brings bad luck.

In mathematics, despite the meaning of the term "Induction", the method is really a deduction method. Each Mathematical Induction proof generally must have three steps:

Assume we want to prove the truth of a statement $S(n)$ for all $n \geq 1$, where $n \in N$.

Step 1: Prove that $S(1)$ holds true, i.e. prove that $S(n)$ holds true when $n = 1$.

Step 2: Assume $S(n)$ holds true for some value of n then prove that $S(n + 1)$ must hold true. [5]

Step 3: Knowing that $S(1)$ holds true, using step 2 we deduce that $S(2)$ holds true; But since $S(2)$ holds true, $S(3)$ must hold true using step 2 again, and so on. Therefore, we can conclude that $S(n)$ holds true for all $n \geq 1$, by the principle of Mathematical Induction.

1.2.1 The Series type

Example 1.5

Prove by Induction that $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all integers $n \geq 1$.

Let $n = 1$, LHS = $1^2 =$ 1, RHS = $\frac{1}{6} \times 1 \times 2 \times 3 = 1$. $\therefore$ It's true for $n = 1$.[6]

Assume that $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for some integer n[7].

Required to prove that $\sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

$$\begin{aligned}
\text{LHS} &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2, \text{ from the assumption,} \\
&= \frac{1}{6}(n+1)\big(n(2n+1) + 6(n+1)\big) \\
&= \frac{1}{6}(n+1)\big(2n^2 + 7n + 6\big) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3), \text{ by factorising,} \\
&= \text{RHS.}
\end{aligned}$$

$\therefore$ The statement holds true for $n + 1$ if it holds true for some integer n.

$\therefore$ By the principle of Mathematical Induction, it's true for all integers $n \geq 1$.

---

[5] Logically equivalent, assume $S(k)$ holds true then prove that $S(k + 1)$ must hold true. The reason is that n or k is some integer. In this step, we only *assume* that the statement is true for some integer, n or k or any letter.

[6] Do not ignore the importance of this step. It is advisable that you clearly show why LHS = RHS.

[7] Using symbols, write 'Assume $\exists n \in N : \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$', i.e. assume there exists an integer n such that the statement holds true.

1.2.2 The Product type

Example 1.6

Prove by Mathematical Induction that $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}, \forall n\geq 2, n\in Z$.

Let $n = 2$, LHS = $1-\frac{1}{4}=\frac{3}{4}$. RHS = $\frac{2+1}{2\times 2}=\frac{3}{4}$. $\therefore$ It's true for $n = 2$.

Assume that $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for some integer n.

Required to prove that $\prod_{k=2}^{n+1}\left(1-\frac{1}{k^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2(n+1)}$.

$$\begin{aligned}
\text{LHS} &= \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{(n+1)^2}\right), \text{ from the assumption,}\\
&= \left(\frac{n+1}{2n}\right)\left(\frac{(n+1)^2-1}{(n+1)^2}\right)\\
&= \left(\frac{n+1}{2n}\right)\left(\frac{n^2+2n}{(n+1)^2}\right)\\
&= \left(\frac{n+1}{2n}\right)\left(\frac{n(n+2)}{(n+1)^2}\right)\\
&= \frac{n+2}{2(n+1)} = \text{RHS.}
\end{aligned}$$

$\therefore$ The statement holds true for $n + 1$ if it holds true for some integer n.

$\therefore$ By the principle of Mathematical Induction, it's true for all integers $n \geq 2$.

1.2.3 The Divisibility type

Example 1.7

Prove that $5^n - 1$ is divisible by 4, for all integers $n \geq 1$

Let $n = 1$, $5 - 1 = 4$, which is obviously divisible by 4. $\therefore$ It is true for $n = 1$.

Assume $5^n - 1$ is divisible by 4 for some integer n, $\therefore 5^n - 1 = 4M$, where M is a positive integer, $\therefore 5^n = 4M + 1$.

Required to prove that $5^{n+1} - 1$ is divisible by 4.

$$\begin{aligned}
\text{Now } 5^{n+1} - 1 &= 5^n \times 5 - 1\\
&= (4M + 1) \times 5 - 1\\
&= 20M + 5 - 1\\
&= 20M + 4\\
&= 4(5M + 1) \text{ which is a multiple of 4.}
\end{aligned}$$

$\therefore$ The statement holds true for $n + 1$ if it holds true for some integer n.

$\therefore$ By the principle of Mathematical Induction, it's true for all integers $n \geq 1$.

1.2.4 The Inequality type

Example 1.8

Prove that $2^n > n^2$ for all integers $n \geq 5$.

Let $n = 5$, LHS = $2^5 = 32$, RHS $= 5^2 = 25$, $\therefore$ LHS > RHS , $\therefore$ It is true for $n = 5$.

Assume that $2^n > n^2$ for some value of n.

Required to prove that $2^{n+1} > (n+1)^2$.

LHS = $2^{n+1} = 2 \times 2^n$

$> 2n^2$, by assumption.

We now need to prove that $2n^2 > (n+1)^2 = n^2 + 2n + 1$, which is equivalent to $n^2 > 2n + 1$, for $n \geq 5$.

Method 1: By Calculus, let $f(x) = x^2 - 2x - 1$ for $x \geq 5, x \in R$.

$f'(x) = 2x - 2 \geq 8 > 0$ for $x \geq 5$, $\therefore f(x)$ is increasing.

But $f(5) = 25 - 11 = 14$, $\therefore f(x) \geq 14 > 0$, i.e. $x^2 - 2x - 1 > 0$ or $x^2 > 2x + 1$ for $x \geq 5$.

Method 2: By graphs, the 2 curves $y = x^2$ and $y = 2x + 1$ meet at $x = 1 \pm \sqrt{2}$.

The points of intersection approximate $(-0.4, 0.17)$ and $(2.4, 5.8)$.

Since $x^2 \geq 2x + 1$ for $x \geq 2.4$, $x^2 > 2x + 1$ for $x \geq 5$.

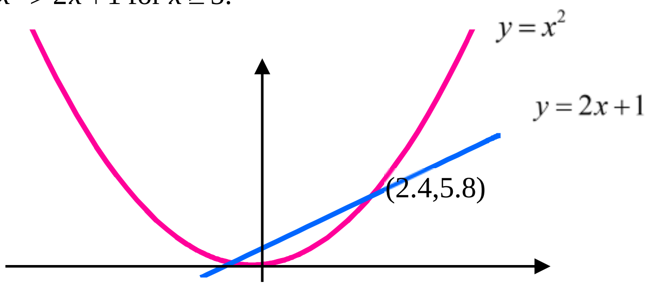

fig. 1.1

$\therefore 2n^2 > (n+1)^2$.

$\therefore$ The statement holds true for $n + 1$ if it holds true for some integer n.

$\therefore$ By the principle of Mathematical Induction, it's true for all integers $n \geq 5$.

1.2.5 The Double Induction type

Example 1.9

Consider the Fibonacci sequence: 1, 1, 2, 3, 5, 8, … which is defined as $T_1 = T_2 = 1$ and $T_{n+2} = T_{n+1} + T_n$.

Prove by Mathematical Induction that $T_n < \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Let $n = 1$, $T_1 = 1 < \frac{7}{4}$; and let $n = 2, T_2 = 1 < \frac{7^2}{4^2} = \frac{49}{16}$. $\therefore$ It is true for both $n = 1$ and 2.

Assume that $T_n < \left(\frac{7}{4}\right)^n$ and $T_{n+1} < \left(\frac{7}{4}\right)^{n+1}$ for some integer n.

Required to prove that $T_{n+2} < \left(\frac{7}{4}\right)^{n+2}$.

$$\text{LHS} = T_{n+2} = T_{n+1} + T_n \text{, by definition of the sequence,}$$

$$< \left(\frac{7}{4}\right)^{n+1} + \left(\frac{7}{4}\right)^{n} \text{, by assumptions,}$$

$$= \left(\frac{7}{4}\right)^{n}\left(\frac{7}{4}+1\right)$$

$$= \left(\frac{7}{4}\right)^{n}\left(\frac{11}{4}\right)$$

$$= \left(\frac{7}{4}\right)^{n}\left(\frac{44}{16}\right)$$

$$< \left(\frac{7}{4}\right)^{n}\left(\frac{49}{16}\right)$$

$$= \left(\frac{7}{4}\right)^{n+2}.$$

$\therefore$ The statement holds true for $n + 2$ if it holds true for some integers n and $n + 1$.

$\therefore$ By the principle of Mathematical Induction, it's true for all integers $n \geq 1$.

1.2.6 Miscellaneous types

Example 1.10

In a room of n people, if everyone has to shake hands with each other once, prove by Mathematical Induction that the number of hand-shakes is $\frac{n(n-1)}{2}$.

If there are two people, obviously there is only one hand-shake.

Substituting $n = 2$ into the formula gives $\frac{2(2-1)}{2} = 1$. $\therefore$ Therefore, the formula holds true for n = 2.

Assume that there are $\frac{n(n-1)}{2}$ handshakes amongst n people.

Required to prove there are $\frac{(n+1)(n)}{2}$ handshakes if there are $n + 1$ people.

Now, suppose a new guest arrives, he (she) must shake hands with everyone already in the room, thus, he (she) must shake n hands.

$$\therefore \text{The total number of hand-shakes} = \frac{n(n-1)}{2} + n$$

$$= \frac{n^2 - n + 2n}{2}$$

$$= \frac{n^2 + n}{2}$$

$$= \frac{(n+1)(n)}{2} \text{, as required.}$$

$\therefore$ By the principle of Mathematical Induction, it's true for all integers $n \geq 2$.

Exercise 1.2

1 Prove by Mathematical Induction for all integers $n \geq 1$.

(a) $1+2+3+\ldots+n=\dfrac{n(n+1)}{2}$.

(b) $1^3+2^3+3^3+\ldots+n^3=\dfrac{n^2(n+1)^2}{4}$.

(c) $1+3+5+\ldots+(2n-1)=n^2$.

(d) $2+4+6+\ldots+2n=n(n+1)$.

(e) $\sum_{k=1}^{n} 2^{k-1}=2^n-1$.

(f) $\sum_{k=1}^{n}(2k-1)^2=\dfrac{n(4n^2-1)}{3}$.

(g) $\sum_{k=1}^{n}\dfrac{1}{k(k+1)}=\dfrac{n}{n+1}$.

(h) $\sum_{k=1}^{n}k(k+1)^2=\dfrac{n(n+1)(n+2)(3n+5)}{12}$.

(i) $\sum_{k=1}^{n}k(k+1)(k+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$.

(j) $\sum_{k=1}^{2n}k=1+2+3+\ldots+(2n)=n(2n+1)$. [8]

(k) $\sum_{k=n}^{2n}k=\dfrac{3n(n+1)}{2}$.

(l) $\sum_{k=n}^{2n}k(k+1)=\dfrac{n(n+1)(7n+5)}{3}$.

(m) $\sum_{k=n}^{n^2}k=n+(n+1)+(n+2)+\ldots+n^2=\dfrac{n(n^3+1)}{2}$. Hint: The term precedes n^2 is n^2-1.

2 Prove by Mathematical Induction for all integers $n \geq 1$, unless stated otherwise.

(a) $\prod_{k=3}^{n}\left(1-\dfrac{2}{k}\right)=\left(1-\dfrac{2}{3}\right)\left(1-\dfrac{2}{4}\right)\left(1-\dfrac{2}{5}\right)\ldots\left(1-\dfrac{2}{n}\right)=\dfrac{2}{n(n-1)}$ for all integers $n \geq 3$.

(b) $\prod_{k=1}^{n}\left(\dfrac{(2k-1)(2k+3)}{(2k+1)^2}\right)=\dfrac{2n+3}{3(2n+1)}$.

(c) $\prod_{k=1}^{n}\left(1+\dfrac{2}{k}\right)=\dfrac{(n+1)(n+2)}{2}$.

(d) $\prod_{k=2}^{n}\left(1-\dfrac{2}{k(k+1)}\right)=\dfrac{n+2}{3n}$.

(e) $\prod_{k=3}^{2n}\left(1-\dfrac{2}{k}\right)=\dfrac{1}{n(2n-1)}$.

(f) $(n+1)(n+2)\ldots(2n)=2^n\left(1\times3\times5\times\ldots(2n-1)\right)$ for all integers $n \geq 2$.

3 Prove by Mathematical Induction for all integers $n \geq 1$, unless stated otherwise.

(a) 6^n+4 is divisible by 10.

(b) 3^{2n} is a multiple of 9.

(c) n^3+2n is divisible by 3.

(d) $5^n+2\times11^n$ is a multiple of 3.

(e) $13\times6^n+2$ is divisible by 10.

(f) $n(n+1)(n+2)$ is a multiple of 6.

(g) 7^n+6^n is divisible by 13 for all odd integers $n \geq 1$.

(h) 3^n+7^n is a multiple of 10 for all odd integers $n \geq 1$.

(i) n^2+2n is divisible by 8 for all even integers $n \geq 2$.

(j) n^3+2n is a multiple of 12 for all even integers $n \geq 2$.

4 Prove by Mathematical Induction for all integers $n \geq 1$, unless stated otherwise.

(a) $3^n \geq 2n+1$.

(b) $n! \geq 2^{n-1}$. (Note: $n!=1\times2\times3\times\ldots\times n$)

(c) $2^n > n(n+1)$ for $n \geq 5$.

(d) $3^n > n(n+1)(n+2)$ for $n \geq 5$.

(e) $3^n > n^3$ for $n \geq 4$.

(f) $3^n > 2n^2$ for $n \geq 2$.

---

[8] In this series, the first term is 1, each successive term is formed by adding 1 to the previous term and the last term is $2n$. Therefore, when $n = 1$, the series produces 1 + 2; when $n = 2$, the series produces 1 + 2 + 3 + 4.

(g) $2^{3^n} > 3^{2^n}$ for $n \geq 2$. (h) $3^{2^n} > 5^{2^n}$ for $n \geq 4$.

5 Prove by Mathematical Induction that $x^n - y^n$ is divisible by $x - y$ for all integers $n \geq 2$.

6 (a) Prove by Mathematical Induction that $(1+x)^n - 1$ is divisible by x for all integers $n \geq 1$.

(b) Factorise $12^n - 4^n - 3^n + 1$. By using the result of part (a), deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all integers $n \geq 1$.

7 (a) Prove by Mathematical Induction that $\sin(x_1 + x_2 + \ldots + x_n) = a_1 \sin x_1 + a_2 \sin x_2 + \ldots + a_n \sin x_n$, where $a_1, a_2, \ldots, a_n$ are real numbers such that $|a_i| \leq 1$ for $i = 1, 2, \ldots, n$.

(b) Hence, deduce that $\sin nx \leq n \sin x$ for $0 \leq x \leq \pi$.

8 (a) Prove that $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$, for $n \geq 1$, by Mathematical Induction.

(b) Prove that $\frac{2n}{3}\sqrt{n} \leq \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}$ by using a graphical means.

(c) Hence, estimate $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + 100$ to the nearest hundred.

9 (a) Prove by Mathematical Induction that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$, for all $n \geq 1$.

(b) Prove that $\frac{1}{n^2} > \frac{1}{n} - \frac{1}{n+1}$, hence, deduce that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} > \frac{3}{2} - \frac{1}{n+1}$, for all $n \geq 1$.

(c) Show that $1.49 < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{99^2} < 1.99$.

10 Let $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} + \frac{1}{2n}$, where $n \geq 1$.

(a) Graphically show that $t_n < \ln 2 < t_n + \frac{1}{2n}$.

(b) Let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$. Prove by Mathematical Induction that $s_n = t_n$.

(c) Hence, find to 3 decimal places, the value of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{9999} - \frac{1}{10\,000}$.

11 A sequence T_n is defined by $T_1 = 5, T_2 = 7$ and $T_{n+2} = 3T_{n+1} - 2T_n$.
Prove by Mathematical Induction for all integers $n \geq 1$ that $T_n = 3 + 2^n$.

12 A sequence T_n is defined by $T_1 = T_2 = 1$ and $T_{n+2} = T_{n+1} + T_n$.

(a) Prove by Mathematical Induction for all $n \geq 1$ that $T_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, where α and $\beta, \alpha > \beta$, are the roots of the equation $x^2 - x - 1 = 0$.

(b) Prove that T_{3n} is always even for all integers $n \geq 1$.

(c) Assume the ratio $\frac{T_{n+1}}{T_n}$ approaches a limit as $n \to \infty$, show that $\frac{T_{n+1}}{T_n} \to \frac{1+\sqrt{5}}{2}$.

13 A sequence S_n is defined by $S_1 = 1, S_2 = 2$ and $S_n = S_{n-1} + (n-1)S_{n-2}$, for $n \geq 2$.

(a) Find S_3 and S_4.

(b) Prove that $\sqrt{x} + x > \sqrt{x(x+1)}$.

(c) Prove by Mathematical Induction that $S_n \geq \sqrt{n!}$ for all integers $n \geq 1$.

14 (a) Prove that if $f(x)=\dfrac{x}{\sqrt{1+x^2}}$ then $f(f(x))=\dfrac{x}{\sqrt{1+2x^2}}$.

(b) Prove by Mathematical Induction that $\underbrace{f(f(\ldots f}_{n\text{ times}}(x)\ldots)=\dfrac{x}{\sqrt{1+nx^2}}$ for all integers $n \ge 2$.

15 (a) If a and b are two positive numbers, explain why $(a-b)(a^n-b^n)\ge 0$.

(b) Hence, prove by Mathematical Induction that $\dfrac{a^n+b^n}{2}\ge\left(\dfrac{a+b}{2}\right)^n$ for all positive integers $n \ge 1$.

16 Prove by Induction that the sum of exterior angles of a n-sided convex polygon is 360°, $n \ge 3$.

17 Given n points in a plane, no three points are collinear, prove by Induction that the number of lines joining any two points is $\dfrac{n(n-1)}{2}$ for all integers $n \ge 2$.

18 Prove by Mathematical Induction that the number of diagonals in a n-sided regular convex polygon is $\dfrac{n(n-3)}{2}$ for all integers $n \ge 3$.

19 Each Mathematical Induction proof involves with 3 steps. Which of the 3 steps does not hold true in the following statements? Justify your answer.

(a) n^2+3n+1 is even for all integers $n \ge 1$. (b) $3^n > n^3$ for all integers $n \ge 1$.

1.3 Inequality proofs

1.3.1 Inequality by Calculus

Example 1.11

Prove that $\dfrac{\ln x}{x}\le\dfrac{1}{e}$. Hence, deduce that $x^e \le e^x$.

Let $f(x)=\dfrac{\ln x}{x}$.

$$f'(x)=\frac{\frac{1}{x}\times x-\ln x}{x^2}=\frac{1-\ln x}{x^2}.$$

$$f''(x)=\frac{-\frac{1}{x}\times x^2-2x(1-\ln x)}{x^4}=\frac{-x-2x+2x\ln x}{x^4}=\frac{2\ln x-3}{x^3}.$$

$f'(x)=0$ when $\ln x=1, \therefore x=e$.

When $x=e$, $f''(e)=\dfrac{-1}{e^3}<0$: $\left(e,\dfrac{1}{e}\right)$ is a maximum point.

$\therefore \dfrac{\ln x}{x}\le\dfrac{1}{e}$.

$\therefore \ln x\le\dfrac{x}{e}$, on multiplying by x, noting $x>0$.

$\therefore x\le e^{\frac{x}{e}}$.

$\therefore x^e\le e^x$.